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THEORETICAL DETERMINATION OF A REFERENCE ENTHALPY FOR THE LAMINAR BOUNDARY LAYER OF THE FLAT PLATE. PRANDTL NUMBER EFFECT

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R. Michel and G. Kretzsachmar

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THEORETICAL DETERMINATION OF A REFERENCE ENTHALPY FOR THE LAMINAR BOUNDARY LAYER OF THE FLAT PLATE. PRANDTL NUMBER EFFECT

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ABSTRACT

A theoretical determination of the reference enthalpy is suggested for the laminar flat plate boundary layer in compressible flow. Crocco's equation is used in evaluating the shear stress within the boundary layer; an approximate solution corresponding to $\rho\mu$ constant is adjusted to satisfy the integral momentum equation.

The reference enthalpy thus depends upon the conditions at the wall and in the external flow and upon coefficients which are functions of Prandtl number. Author

NOMENCLATURE

$C = \frac{\rho^*\mu^*}{\rho_e\mu_e}$

$C_f = \frac{\tau_p}{\rho_e u_e^2/2}$, friction coefficient

C_{fi} : friction coefficient for incompressible fluids

$C_h = \frac{\varphi_p}{\rho_e u_e (h_p - h_f)}$, coefficient of heat flux

C_p : specific heat at constant pressure

$F = \frac{\tau}{\rho_e u_e^2/2}\left(\frac{\mu_e}{\rho_e u_e x}\right)^{1/2}$, dimensionless friction

F_i : dimensionless friction for incompressible fluids

h : enthalpy

h_f : enthalpy of friction

h_i : enthalpy at rest

$i = \frac{h}{h_e}$, dimensionless enthalpy

M : Mach number
τ : Prandtl number

R_x $= \frac{\rho_e u_e x}{\mu_e}$, Reynolds number

r $= \frac{h_f - h_e}{h_{ie} - h_e}$, recovery factor

s $= \frac{C_h}{C_f/2}$, similarity ratio

T : temperature
T_f : friction temperature

u : longitudinal component of the velocity

w $= \frac{u}{u_e}$, dimensionless velocity

x, y : abscissa and ordinate of a cartesian system of axes
δ : thickness of the boundary layer

δ_2 $= \int_0^{\delta} \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$, momentum thickness

η $= \frac{y}{x}\sqrt{R_x}$, dimensionless ordinate

θ^{I}, θ^{II} : functions of w and τ involved in the enthalpy and velocity relation
λ : coefficient of thermal conductivity
μ : viscosity coefficient
ρ : mass per unit volume

τ $= \mu \frac{\partial u}{\partial y}$, friction

φ $= -\lambda \frac{\partial T}{\partial y}$, heat flux

e : subscript referring to the outer boundary of the layer /4
p : subscript referring to the value at the skin
* : superscript referring to the value at the reference temperature

I. Introduction

Exact mathematical solutions to the laminar boundary layer equations exist for the flat plate and compressible fluids. Their order of approximation depends only upon the assumptions made concerning the physical properties of the gas. For example the solution of Crocco and Van Driest, which makes use of Sutherland's viscosity and temperature relation, and which involves a constant Prandtl number different from one, must lead to very accurate results within very wide ranges of Mach numbers and skin temperatures.

The concept of reference enthalpy plays an essential role in yielding simple equations for the essential characteristics of the boundary layer, and initially for the skin friction. This concept was originally introduced in an empirical way, by adjusting the coefficients involved in the reference enthalpy, to yield the best possible agreement with the selected solution.

Certain attempts at less empirical determinations were made. Monaghan proposed, for example, to take for the reference enthalpy an average value of the enthalpy with respect to the velocity within the boundary layer.

It must be noted, however, that the reference enthalpy can only be defined in terms of the proposed goal. If the property investigated is skin friction, it seems logical to attempt to determine the reference enthalpy starting from an equation describing the behavior of the friction inside the boundary layer. On this idea is based the determination of the reference enthalpy which is proposed in this article. The equation employed will be that of Crocco, in order to describe the behavior of the friction with respect to the velocity within the boundary layer. An approximate solution to this equation will lead, for the friction coefficient, to the usual form which is offered by the concept of reference enthalpy. The reference enthalpy will itself be determined by demanding for the solution an additional condition, namely: that it must satisfy the general momentum equation.

II. The Concept of Reference Enthalpy

The application of the reference enthalpy concept to the skin friction consists in extending the relation which gives, for incompressible fluids, the coefficient of local friction as a function of the Reynolds number of the abscissa. Equation (1) can be employed in incompressible fluids provided we take the values ρ^* and μ^* which correspond to a certain reference enthalpy h* to be selected from certain flow parameters of the boundary layer under consideration.

First let us examine the example of the laminar boundary layer for the flat plate. For the incompressible fluids the friction equation was

$$\frac{\tau_p}{\rho u_e^2/2} = \frac{0,664}{(\rho u_e x/\mu)^{1/2}}.$$

Typo R
Proof ...
Check ...

For the compressible case we have:

$$\frac{\tau_p}{\rho^* u_e^2/2} = \frac{0{,}664}{(\rho^* u_e x/\mu^*)^{1/2}}.$$

The friction coefficient C_f however is obtained by having τ_p in terms of $\rho_e u_e^2/2$ and we seek to express it as a function of the Reynolds number obtained from the exterior values, $R = \frac{\rho_e u_e x}{\mu e}$.

The concept of reference enthalpy gives in this way

$$\boxed{\frac{C_f}{C_{fi}} = \sqrt{\frac{\rho^*\mu^*}{\rho_e\mu_e}}}. \qquad (1)$$

for the ratio of the friction coefficient to the coefficient C_{fi} in the compressible case, for the same value of the Reynolds number.

The problem then consists in determining the value of that product $\rho^*\mu^*$. The reference enthalpy is simply the enthalpy h* corresponding to this product.

The interpretation of theoretical or experimental results has led to various relations for the reference enthalpy. This interpretation has often been made in cases where the specific heat could be taken as constant, and a reference temperature in this case was used instead of an enthalpy.

Generally speaking, the three parameters which are used for determining the reference enthalpy are:

1. the skin enthalpy
2. the enthalpy h_e, or the enthalpy of rest h_{ie} of the outer flow
3. the enthalpy of friction or enthalpy of the athermanous skin. It is known that it can be related to h_e and h_{ie} by means of the recovery factor r;

$$h_f - h_e = r\ (h_{ie} - h_e).$$

From Eckert (Ref. 1) the reference enthalpy is given by the relation

$$h^* - h_e = 0.50\ (h_p - h_e) + 0.22\ (h_f - h_e); \qquad (2a)$$

This formula was first proposed for the laminar boundary layer. According to that author, however, it can also give good results for the turbulent case.

Monaghan (Ref. 2) proposed for the reference enthalpy a value of the enthalpy averaged over the velocity, from an approximation to the relation that can exist in the boundary layer between the enthalpy and the velocity. This relation is almost the same for the laminar boundary layer as for the turbulent boundary layer; the same reference enthalpy is obtained for both cases:

$$h^* - h_e = 0.54\ (h_p - h_e) + 0.16\ (h_f - h_e). \tag{2b}$$

Sommer and Short (Ref. 3) found from their experiments with a turbulent boundary layer a relation which can be put into the preceding form and which is written as follows, assuming a recovery factor of 0.9:

$$h^* - h_e = 0.45\ (h_p - h_e) + 0.195\ (h_f - h_e). \tag{2c}$$

We see that in these relations the reference enthalpy results from a linear combination of the enthalpy deviation $(h_p - h_e)$ and the enthalpy deviation $(h_f - h_e)$. Note that when $h_p = h_f$ (athermanous skin), $(h^* - h_e)$ becomes proportional to $(h_f - h_e)$, the proportionality coefficient being 0.72 from Eckert, 0.70 from Monaghan, 0.65 from Sommer and Short. These are small differences that lead to close results. A comparison with exact theories seems to slightly favor the relation of Monaghan. /5

III. Review of the Results of the Crocco-Van Driest Solution

Among the theoretical solutions available for the flat plate in compressible fluids that given by Crocco is one of the most exact. It employs a constant Prandtl number, and the energy equation is written down for the enthalpy. Numerical solutions have been calculated by Crocco (Ref. 4) and then by Van Driest (Ref. 5) with the help of Sutherland's law of viscosity vs temperature. It can be applied to the case where the wall temperature is constant.

We know that Crocco's transformation consists in writing local equations for the momentum and the energy, taking x and u, instead of x and y, as the independent variables, and taking the friction τ between the streams, and the enthalpy h as the dependent variables. An assumed affinity transformation permits us to go from equations with partial derivatives to simple differential equations with the velocity u being the variable. In a nondimensional form we have the transformed variables which are involved in Crocco's equation:

$$F = \frac{\tau}{\rho_e u_e^2/2}\left(\frac{\rho_e u_e x}{\mu_e}\right)^{1/2};$$
$$i = \frac{h}{h_e};$$
$$w = \frac{u}{u_e}.$$

The local equations for the momentum and the energy are written, respectively:

$$FF'' + 2\frac{\rho\mu}{\rho_e\mu_e}w = 0; \tag{3}$$
$$i'' + (1-\sigma)\frac{F'}{F}i' + \sigma\frac{u_e^2}{h_e} = 0; \tag{4}$$

the derivative being taken with respect to the variable w.

Integrating this system of equations leads essentially to the following results:

1. From Equation (3) we obtain the distribution of the internal friction within the boundary layer as a function of the velocity. An essential characteristic is of course the value of F at $w = 0$, in other words at the skin:

$$F_p = C_f\sqrt{\mathcal{R}_x}.$$

From the distribution of the friction as a function of w we then deduce the distribution of the velocity as a function of y, observing that $\tau = \mu\frac{\partial u}{\partial y}$. In this way we obtain for the usual variable

$$\eta = \frac{y}{x}\sqrt{\mathcal{R}_x} = 2\int_0^w \frac{\mu/\mu_e}{F}\,dw.$$

The distributions of friction in the boundary layer have been calculated by Crocco and Van Driest, for a wide range of Mach numbers and skin temperatures. The results which were obtained show that when the friction is divided by its value at the skin the distribution $\frac{\tau}{\tau_p}$ (w) is little affected by the compressibility and remains close to that of the incompressible fluid. This property is used in the integration of the energy equation, where Van Driest retains the friction distribution given by Blasius's solution.

2. The function of the enthalpy versus velocity within the boundary layer is obtained from equation (4). The following relation results:

$$h = h_p - (h_p - h_e)\,\theta^{I} + u_e^2\theta^{II} \tag{5a}$$

where θ^{I} and θ^{II} are functions of $w = \frac{u}{u_e}$ calculated and given in reference 5 for different values of the Prandtl number. It is useful in what follows to use the enthalpy of friction $h_f = h_e + r\frac{u_e^2}{2}$ in the enthalpy vs velocity relation. The relation becomes:

$$h - h_e = (h_p - h_e)\,(1 - \theta^{I}) + (h_f - h_e)\,\frac{2\theta^{II}}{r}. \tag{5b}$$

Finally, the enthalpy vs velocity relation gives (from the slope at the skin) the two following essential parameters:

1. the recovery factor r
2. the analogy factor s, which is the ratio of the heat flux to the friction coefficient at the skin, $s = \frac{C_h}{C_f/2}$

These two factors depend only on the Prandtl number. They are given in Table 1, and they obey fairly closely the relations:

$$r = \tau^{1/2}; \qquad s = \tau^{-2/3}.$$

IV. Approximate Solution to Crocco's Equation. Corresponding Reference Enthalpy

An approximate solution to Crocco's equation for the friction can be sought by taking for the ratio $\frac{\rho\mu}{\rho_e\mu_e}$ a constant average value, to be chosen somewhere within the boundary layer.

First assume that μ is proportional to T. From the equation of state μ is then inversely proportional to ρ and $\frac{\rho\mu}{\rho_e\mu_e} = 1$.

Consequently we again find the form of the equation for incompressible fluids:

$$F_i F_i'' + 2\,w = 0.$$

This corresponds to Blasius's equation, whose solution is well known and gives for the skin friction

$$F_{ip} = C_{fi}\ \sqrt{R_x} = 0.664.$$

We now take for $\frac{\rho\mu}{\rho_e\mu_e}$ an average value $\frac{\rho^*\mu^*}{\rho_e\mu_e} = C$. We have

$$FF'' + 2\,Cw = 0.$$

The solution to this equation is immediately deduced from that of incompressible fluids through the relation

$$F = F_i\ \sqrt{C};$$

or taking the values at the skin,

$$C_f\sqrt{\mathcal{R}_x} = C_{fi}\sqrt{\mathcal{R}_x}\sqrt{C} = 0{,}664\sqrt{\frac{\rho^*\mu^*}{\rho_e\mu_e}}.$$

We again find the relation (1) exactly, by using the concept of reference enthalpy. This concept can therefore be considered as corresponding to an approximate solution to the flat plate boundary layer equations, with $\rho\mu$ being constant. It remains now to determine how the average values $\rho^*\mu^*$ must be chosen. /6

In order to arrive at this we shall demand that the solution obey not only the approximate local equation, but also the overall equation for the momentum (Karman's equation)

$$C_f = 2\frac{d\delta_2}{dx}, \qquad \text{where} \qquad \delta_2 = \int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy;$$

We shall first write this equation with the new variables. Observing that:

$$dy = 2\sqrt{\frac{\mu_e}{\rho_e u_e x}}\ x\ \frac{\mu/u_e}{F}\,dw$$

the momentum thickness is written

$$\delta_2 = 2\sqrt{\frac{\mu_e}{\rho_e u_e x}}\ x\int_0^1 \frac{\rho\mu}{\rho_e\mu_e}\frac{w\,(1-w)}{F}\,dw$$

Taking the derivative of δ_2 with respect to x, Karman's equation takes the form:

$$C_f \sqrt{\mathcal{R}_x} = 2 \int_0^1 \frac{\rho\mu}{\rho_e\mu_e} \frac{w(1-w)}{F} dw.$$

Now replace $C_f \sqrt{R_x}$ and F by the results of the solution for constant $\rho\mu$

$$F = F_i \sqrt{\frac{\rho^*\mu^*}{\rho_e\mu_e}}; \qquad C_f \sqrt{\mathcal{R}_x} = 0{,}664 \sqrt{\frac{\rho^*\mu^*}{\rho_e\mu_e}};$$

Karman's equation gives:

$$\frac{\rho^*\mu^*}{\rho_e\mu_e} = \frac{1}{0{,}332} \int_0^1 \frac{\rho\mu}{\rho_e\mu_e} \frac{w(1-w)}{F_i} dw. \tag{6a}$$

This equation determines how the average value $\frac{\rho^*\mu^*}{\rho_e\mu_e}$ of the approximate solution with constant $\rho\mu$ must be chosen. The friction distribution F_i of the incompressible fluid is a known function of $w = \frac{u}{u_e}$ and it is given by Blasius's equation. It is useful to verify that Blasius's solution for the incompressible fluid gives exactly:

$$\int_0^1 \frac{w(1-w)}{F_i} dw = 0{,}332.$$

The previous relation can be put into the form

$$\frac{\rho^*\mu^*}{\rho_e\mu_e} - 1 = \frac{1}{0{,}332} \int_0^1 \left(\frac{\rho\mu}{\rho_e\mu_e} - 1\right) \frac{w(1-w)}{F_i} dw. \tag{6b}$$

The reference enthalpy has the value h* of the enthalpy which corresponds to the preceding value of $\rho^*\mu^*$. It would be desirable to add to (6b) a relation as exact as possible between the viscosity and the temperature, e.g., Sutherland's law. It is obvious however that Sutherland's law would lead (for the reference enthalpy) to an expression much more complicated than the linear relations for the parameters h_p, h_e, h_f presented in Section II.

A linear enthalpy can only be obtained if a linear approximation is chosen for the changes of the product $\rho\mu$ with the enthalpy. This is the assumption we shall now make, and we shall take that $\frac{\rho\mu}{\rho_e\mu_e} - 1$ is proportional to $\frac{h}{h_e} - 1$ in order to retrieve $\rho\mu = \rho_e\mu_e$ at $h = h_e$. The relation

(6b) becomes:

$$h^* - h_e = \frac{1}{0{,}332} \int_0^1 (h - h_e) \frac{w(1-w)}{F_i} dw. \qquad (7)$$

Now equation (5b), giving the changes of h with w, must be used to obtain the reference enthalpy. Integrating, we obtain:

$$\boxed{h^* - h_e = A(h_p - h_e) + B(h_f - h_e);} \qquad (8)$$

with

$$A = \frac{1}{0{,}332} \int_0^1 (1 - \theta^I) \frac{w(1-w)}{F_i} dw; \qquad (9a)$$

$$B = \frac{2}{0{,}332} \int_0^1 \frac{\theta^{II}}{r} \frac{w(1-w)}{F_i} dw. \qquad (9b)$$

The form which is obtained here is identical with that of the empirical relations of Eckert and Monaghan.

We shall now recall that $F_i(w)$ is the friction distribution for the incompressible fluid as given by the solution of Blasius, and that θ^I and θ^{II} are functions of $w = \frac{u}{u_e}$ calculated and given by Van Driest for different values of the Prandtl number. The coefficients in the relation for the reference enthalpy depend therefore on the Prandtl number.

The integration corresponding to the formulas of (9) lead to values of A and B which are listed in Table 1 and are shown in the curves of Figure 1. For example we have, for $\tau = 0.725$:

$$A = 0.474; \qquad B = 0.178;$$

Table 1. Coefficients A and B of the relation for the reference enthalpy, and recovery and analogy factors as functions of Prandtl number /7

σ	0,5	0,725	0,75	1	1,25	1,50	2
A	0,527	0.474	0,469	0,428	0,396	0,370	0,330
B	0,170	0.178	0,179	0,184	0,187	0.188	0.188
r	0,704	0,851	0,865	1,000	1,118	1,224	1,413
s	1,562	1,230	1,204	1,000	0,865	0.768	0,636

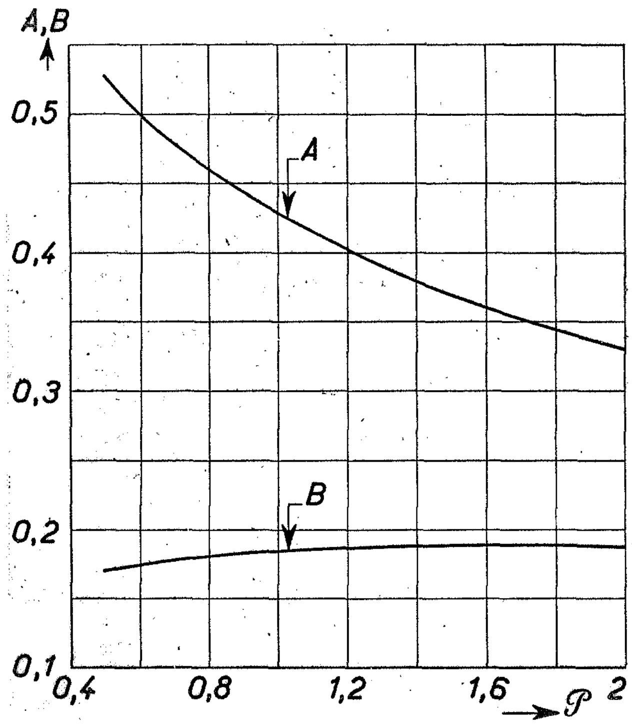

Figure 1. Coefficients of the reference enthalpy as a function of Prandtl number

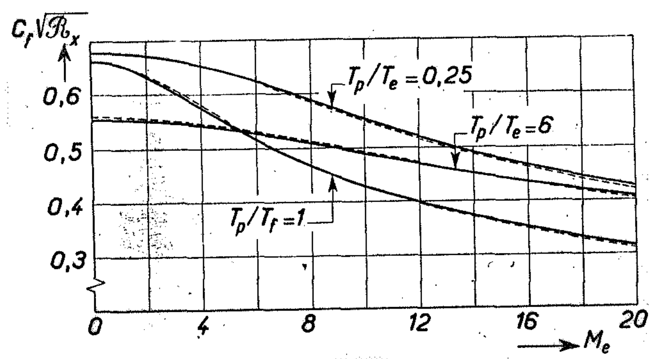

Figure 2. Comparison of the results obtained with the concept of reference enthalpy with the exact results of Crocco and Van Driest

$\tau = 0.75$ $\gamma = 1.4$ $T_e = 218^\circ$ K

———— Crocco - Van Driest

-------- The reference enthalpy as proposed

These coefficients are sufficiently close to those previously obtained empirically.

To determine the accuracy of the results obtained with the preceding method the coefficient of local friction for the flat plate with a laminar boundary layer has been calculated from formula (1) using Sutherland's relation for the relation between the viscosity and the temperature.

The reference enthalpy has been determined by means of formula (8), with the coefficients A and B taken for the Prandtl number equal to 0.75, a value which corresponds to the results of the exact solution (Ref. 5). A comparison of Figure 2 for different values of the skin temperature shows that the friction coefficients are very close to those given by the solution of Crocco and Van Driest.

Conclusions

It is possible to define a reference enthalpy for the laminar boundary layer of the flat plate and to determine numerically the coefficients of the law which relates this enthalpy with the parameters h_p, h_e, h_f.

This can be done from a solution to Crocco's equation with $\rho\mu$ constant, by using the overall momentum equations.

Since the determination of the coefficients involve a relation between the enthalpy and the velocity which depends on the Prandtl number, the reference enthalpy depends also on the Prandtl number. In this way we have available, together with the recovery factor and the analogy factor, a set of results of very wide range which must be capable of predicting the skin friction for gases or mixtures of gases having different physical characteristics.

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